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# Towards UV finite quantum field theories from non-local field operators

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**Abstract.** A non-local toy model whose interaction consists of smeared, non-local field operators is presented. We work out the Feynman rules and propose a power counting formula for arbitrary graphs. Explicit calculations for one loop graphs show that their contribution is finite for sufficient smearing and agree with the power counting formula. UV/IR mixing does not occur.

## 1 Introduction

Already in [1] the behaviour of non-local field theories has been studied, and it has been questioned whether they help to avoid divergences. In a different approach [2], the construction of finite field theories starting from smeared propagators has been investigated. The smeared propagators are considered as a result of gravitational fluctuations at the Planck scale. We want to follow this line of thought and restrict ourselves to scalar field theory. But our starting point is a non-local deformation of the field operators in the interaction, leaving the free Hamiltonian untouched.

Beside the above mentioned approaches to non-local field theories, we want to address four other ones and to distinguish them clearly from our point of view presented here.

In the first approach [3], the point-wise multiplication of scalar fields in the Lagrangian is replaced by a non-local $*$-product. The $*$-product is such that quadratic terms are unaltered,

$$\int \mathrm{d}^4x\, A * B = \int \mathrm{d}^4x\, AB.$$

The Feynman rules are obtained directly from the classical action. Therefore, propagators are unchanged. The only modifications are due to the vertex contributions,

$$\Gamma_4 = \int \mathrm{d}^4x\,(\phi * \phi * \phi * \phi)(x). \qquad (1)$$

Each vertex contributes, beside the coupling constant, a phase factor. In momentum space we get

$$\tilde{V}(k_1,\ldots,k_n) = \delta(k_1+\ldots+k_n)\exp\left(-\frac{\mathrm{i}}{2}\sum_{i<j}^{n} k_{i\mu}k_{j\nu}\sigma^{\mu\nu}\right),$$

where $k_1,\ldots,k_n$ are the incoming momenta, and $\sigma^{\mu\nu}$ is the real antisymmetric deformation parameter of dimension $[\text{length}]^2$

$$[q^\mu, q^\nu] = \mathrm{i}\,\sigma^{\mu\nu}.$$

The drawbacks of this theory are the so-called UV/IR mixing [4] and its non-unitarity [5].

Unitarity can be restored by considering the Hamiltonian instead of the Lagrangian and by computing Feynman rules using the Gell-Mann–Low formula (28) [6–11]. These methods represent a second possibility to describe non-commutative quantum field theories perturbatively. The free propagator is unchanged due to the remarkable fact that the deformed free Hamiltonian $H_0^*$ is equal to the undeformed one,

$$\begin{aligned} H_0^* &= \int \mathrm{d}^3x\,\left(\sum_\mu \partial_\mu\phi * \partial_\mu\phi + m^2\phi*\phi\right) \\ &= \int \mathrm{d}^3x\,\left(\sum_\mu \partial_\mu\phi\,\partial_\mu\phi + m^2\phi\,\phi\right) = H_0, \end{aligned} \qquad (2)$$

with $\phi$ representing free field operators. We will elaborate on this important statement in more detail and generality elsewhere.

The third approach is based on an oscillator representation of non-commutative space-time [12–14]. Let us focus on the presentation given in [14]. Scalar field theory in $D = 2+1$ dimensions is considered. The time component seems artificial. In this sense, the results obtained in [12,13] for 4-dimensional Euclidean space agree, corresponding to $D = 4+1$-dimensional Minkowski space according to [14].

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In [14], time commutes with the spatial coordinates which satisfy the relation

$$\left[\hat{x}^i, \hat{x}^j\right] = \mathrm{i}\theta\epsilon^{ij}, \tag{3}$$

$i, j = 1, 2$. Further on, there are the usual commutation relations with the momenta,

$$\left[\hat{x}^i, \hat{p}_j\right] = \mathrm{i}\delta^i_j\,, \qquad [\hat{p}_i, \hat{p}_j] = 0. \tag{4}$$

New coordinates $\hat{z}$ and $\hat{z}^\dagger$ are introduced [15],

$$\hat{z} = \frac{1}{\sqrt{2}}\left(\hat{x}^1 + \mathrm{i}\hat{x}^2\right), \tag{5}$$
$$\hat{z}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{x}^1 - \mathrm{i}\hat{x}^2\right),$$

in order to obtain

$$\left[\hat{z}, \hat{z}^\dagger\right] = \theta. \tag{6}$$

$\hat{z}$ and $\hat{z}^\dagger$ can be established as annihilation and creation operators of a harmonic oscillator, and coherent states can be used as a basis of the Fock space. Coherent states $|z\rangle$ are eigenstates of the annihilation operator,

$$\hat{z}|z\rangle = z|z\rangle, \quad \langle z|\hat{z}^\dagger = \bar{z}\langle z|. \tag{7}$$

They are given by

$$|z\rangle = \exp\left(-\frac{z\bar{z}}{2\theta} - \frac{z}{\theta}\hat{z}^\dagger\right)|0\rangle, \tag{8}$$

satisfying the completeness relation

$$\frac{1}{\pi\theta}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,|z\rangle\langle z| = 1. \tag{9}$$

Coherent states are not orthogonal, however,

$$\langle w|z\rangle = \exp\left(-\frac{|z|^2 + |w|^2}{2\theta} - \frac{\bar{w}z}{\theta}\right). \tag{10}$$

Via expectation values, one can assign ordinary functions to any operator $F(\hat{x}^1, \hat{x}^2)$,

$$F(z) := \langle z|F\left(\hat{x}^1, \hat{x}^2\right)|z\,\rangle. \tag{11}$$

The algebraic structure of the non-commutative algebra (3) is properly taken care of, i.e.

$$\langle z|\left[\hat{x}^1, \hat{x}^2\right]|z\rangle = \mathrm{i}\theta. \tag{12}$$

With the expansion of a real scalar free field operator

$$\phi(t, z) = \int \tfrac{\mathrm{d}^2 p}{2\pi}\, b_p \exp\left(-\mathrm{i}Et\right)\langle z|\exp\left(\mathrm{i}p_j\hat{x}^j\right)|z\rangle + \text{h.c.},$$
$$\left(\Box_x + m^2\right)\phi(t, x) = 0, \tag{13}$$

the propagator – defined as the expectation value of a time ordered product of field operators – becomes

$$\begin{aligned} G(t_1 - t_2, z_1 - z_2) &= \langle 0|T\phi(t_1, z_1)\phi(t_2, z_2)|0\rangle \\ &= \int \frac{\mathrm{d}E\,\mathrm{d}^2 p}{(2\pi)^{3/2}}\,\frac{-1}{E^2 - \mathbf{p}^{\,2} - m^2}\,\exp\left(-\frac{\theta}{2}\mathbf{p}^{\,2}\right) \\ &\quad\times \exp(-\mathrm{i}E(t_1 - t_2)) \\ &\quad\times \exp\left(\mathrm{i}\frac{p_1}{\sqrt{2}}\left(z_1 - z_2 + \bar{z}_1 - \bar{z}_2\right)\right. \\ &\qquad\left. + \mathrm{i}\frac{p_2}{\sqrt{2}}\left(z_1 - z_2 - \bar{z}_1 + \bar{z}_2\right)\right). \end{aligned} \tag{14}$$

This propagator is the "Green's function" of the ordinary Klein–Gordon equation, with the exception that the delta function is replaced by an approximate (smeared) delta function,

$$\begin{aligned} &\left(\Box_1 + m^2\right) G\left(t_1 - t_2, z_1 - z_2\right) \\ &= \left(-\partial^2_{t_1} + 2\partial_{z_1}\partial_{\bar{z}_1} + m^2\right) G\left(t_1 - t_2, z_1 - z_2\right) \\ &= \frac{2\pi\delta(t_1 - t_2)}{\theta} \\ &\quad\times \exp\left(-\frac{1}{4\theta}\left(z_1 - z_2 + \bar{z}_1 - \bar{z}_2\right)^2\right. \\ &\qquad\left. + \frac{1}{4\theta}\left(z_1 - z_2 - \bar{z}_1 + \bar{z}_2\right)^2\right). \end{aligned} \tag{15}$$

In this case, the free propagator is modified. It experiences an exponential damping (14). It is important to note that the non-commutativity is related to exponentially damped propagators. This fact motivates our model.

In the fourth approach [16], also only the interaction Hamiltonian is modified. The fields are smeared over space-time in the following way:

$$\begin{aligned} &H^*_I(t) = \\ &\lambda c_n \int \mathrm{d}^3 x \int_{\mathbb{R}^{4n}} \mathrm{d}a_1 \ldots \mathrm{d}a_n : \phi(x + a_1)\ldots\phi(x + a_n): \\ &\quad\times \exp\left(-\frac{1}{2}\sum_{j,\mu} a^{\mu\,2}_j\right)\delta^{(4)}\left(\frac{1}{n}\sum_{j=1}^n a_j\right). \end{aligned} \tag{16}$$

Using this ansatz, it has been shown that the Dyson expansion of the S-matrix is finite, order by order.

Similar to the second and fourth approach above, we consider only modifications in the interaction. We replace the local field operators $\phi$ by smeared, non-local fields $\phi_M$, as discussed in the next section. Therefore, the free propagators are not modified. Internal lines, however, will be modified by an exponential damping factor, similar to the third approach. Let us emphasise the difference again: in the third approach, the free propagator is damped, whereas our model possesses ordinary free propagators, but damped internal lines.

In Sect. 3, we will consider 1-loop corrections in order to extend the classical theory. We will see that these contributions are finite.

## 2 Smeared field operators

We want to study the effect of replacing the scalar field operators $\phi(x)$ by blurred operators, smeared over space-time:

$$\phi_M(x) \equiv N \int \mathrm{d}^n a\, \mathrm{e}^{-a^{\mathrm{T}} a}\, \phi(x + Ma), \tag{17}$$

where $a$ is a real Euclidean $n$-dimensional vector, $M$ is a real $4 \times n$ matrix. $N$ denotes a normalisation constant. The integration parameters $a^i$ are assumed to be dimensionless. Therefore, the matrix elements of $M$ have dimension of length. The non-vanishing matrix $M$ generates the non-locality. We will denote Minkowski indices by Greek letters, Euclidean indices by Roman letters. Therefore, the index structure of $M$ is $M^{\mu}{}_{i}$. However, the case $n > 4$ can be reduced to the case $n = 4$. Due to the $QR$-decomposition, the matrix $M$ can be written as a product of the $4 \times n$ matrix $\tilde{R}$ and an orthogonal $n \times n$ matrix $Q$. The first 4 rows of $\tilde{R}$ contain a lower triangular $4 \times 4$ matrix $R$, and all other entries are zero,

$$M = \left[\, R\ 0 \,\right] Q \equiv \tilde{R} Q. \tag{18}$$

The orthogonal matrix $Q$ can be absorbed in an integral transformation, $\tilde{a} = Q\, a$, and we get

$$\phi_M(x) = N \int \mathrm{d}^n \tilde{a}\, \mathrm{e}^{-\tilde{a}^{\mathrm{T}} \tilde{a}}\, \phi\left(x + \tilde{R}\, \tilde{a}\right). \tag{19}$$

Since $\tilde{R}$ has the form shown in (18), the integration over the variables $\tilde{a}_5, \ldots, \tilde{a}_n$ are Gaußian integrals which merely redefine the normalisation constant. Hence, only 4 dimensions are left. From now on, we will stick to that case.

Since the newly defined field operators $\phi_M(x)$ are superpositions of the operators $\phi(x)$, we demand that they are solutions of the free Klein–Gordon equation,

$$\left(\Box_x + m^2\right) \phi_M(x) = 0. \tag{20}$$

The Fourier transform is given by

$$\phi(x + Ma) = \int \frac{\mathrm{d}^4 k}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}k(x+Ma)} \tilde{\phi}(k). \tag{21}$$

Due to the Klein–Gordon equation, we can find a nice expression for the smeared field operators $\phi_M(x)$,

$$\begin{aligned} &\phi_M(x) \\ &= (2\pi)^{-3/2} N \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega_p}} \left[ b(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}p^+ x} + b^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}p^+ x} \right] \\ &\quad \times \int \mathrm{d}^4 a\, \mathrm{e}^{-a^r a^r + \mathrm{i}p^+_\mu M^\mu{}_r a^r} \\ &= (2\pi)^{-3/2} \pi^2 N \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega_p}} \left[ b(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}p^+ x} + b^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}p^+ x} \right] \\ &\quad \times \exp\left( -\frac{1}{4} p^+_\mu p^+_\nu \kappa^{\mu\nu} \right), \end{aligned} \tag{22}$$

where $p^+_\mu = (+\omega_p, -\mathbf{p})$ with $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$. $b$ and $b^\dagger$ obey the canonical commutation relations

$$\left[ b(\mathbf{p})\,,\, b^\dagger(\mathbf{q}) \right] = \delta^3(\mathbf{p} - \mathbf{q}).$$

Summation over repeated indices is implied. Furthermore, we have used the definition

$$\kappa^{\mu\nu} \equiv M^\mu{}_r M^\nu{}_r = \left( M M^{\mathrm{T}} \right)^{\mu\nu}. \tag{23}$$

The matrix $\kappa$ is symmetric. For real $M$, its eigenvalues are always bigger than or equal to zero, i.e. $\kappa$ is positive semidefinite. The exponential factor in (22) is always damping,

$$\exp\left( -\frac{1}{4} p^+_\mu p^+_\nu \kappa^{\mu\nu} \right) \leq 1.$$

As we will see below, $\kappa^{\mu\nu}$ characterises the perturbation theory, not $M$ itself. Therefore, we only have to choose an appropriate matrix $\kappa^{\mu\nu}$ in order to do perturbation theory, ensuring that the matrix can be reproduced by $MM^{\mathrm{T}}$. A tempting choice is $\kappa^{\mu\nu} \propto g^{\mu\nu}$, but $g$ is neither positive nor negative semidefinite. The choice $\kappa = 0$ reproduces local field theory.

We want to study the perturbative quantisation of this kind of deformation, according to the results presented in [11]. The deformed Hamiltonian is defined as

$$H^* = H_0 + V^*, \tag{24}$$

where $H_0$ denotes the free undeformed Hamiltonian of the theory. We have replaced the scalar fields by the smeared fields (17), $\phi \to \phi_M$ in the interaction part of the Hamiltonian only. The free Hamiltonian, $H_0$ is unaltered. Of course, it would be more natural to deform $H_0 \to H_0^*$ also. Then the applicability of the perturbation theory elaborated in [11] is related to the question whether $H_0^* = H_0$ is true or not. If $H_0^* \neq H_0$, we have to define the interaction Hamiltonian as $\tilde{V} = V^* + (H_0^* - H_0)$. In this case, we also have to make sure that the time dependence of $\tilde{V}$ is given by

$$\tilde{V}(t) = \mathrm{e}^{\mathrm{i}H_0 t}\, \tilde{V}(0)\, \mathrm{e}^{-\mathrm{i}H_0\ t}, \tag{25}$$

and the asymptotic behaviour is still governed by $H_0$ and not $H_0^*$.

Let us examine perturbation theory arising from (24), leaving the free Hamiltonian $H_0$ undeformed. The interaction corresponding to $\phi^k$ is deformed as follows:

$$\begin{aligned} V^*\left(x^0\right) &\equiv \frac{\lambda}{k!} \int \mathrm{d}^3 x\, \phi_M^k(x) \\ &= \frac{\lambda}{k!} N^k \int \mathrm{d}^3 x \int \mathrm{d}^4 a_1 \ldots \mathrm{d}^4 a_k \\ &\quad \times \mathrm{e}^{-\sum_i a_i^{\mathrm{T}} a_i}\, \phi(x + Ma_1) \ldots \phi(x + Ma_k). \end{aligned} \tag{26}$$

This is obviously translation invariant. Therefore, we will first relate (26) to the notation introduced in [11] in order to apply the momentum space rules given there, for a general non-local interaction. The interaction has the general form

$$\begin{aligned} V\left(z^0\right) = \int \mathrm{d}^3 z \int \mathrm{d}\underline{\mu}\ w\left(\underline{\mu}\right)\ \phi\left(z + h_1\left(\underline{\mu}\right)\right) \\ \cdot \ldots \cdot \phi\left(z + h_k\left(\underline{\mu}\right)\right), \end{aligned} \tag{27}$$

where

$$\underline{\mu} = \left(a_1^1, a_1^2, a_1^3, a_1^4, a_2^1, \ldots a_k^3, a_k^4\right),$$
$$w\left(\underline{\mu}\right) = \mathrm{e}^{-\sum_{j=1}^k a_j^{\mathrm{T}} a_j},$$
$$h_s(\mu) = M \cdot a_s,\ s = 1, \ldots, k.$$

Following the procedure presented in [11], we obtain the Feynman rules evaluating the Gell-Mann–Low formula:

$$\begin{aligned} &\langle 0|T\phi(x_1)\ldots\phi(x_k)|0\rangle_H \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_m \\ &\quad \times \langle 0|T\phi(x_1)\ldots\phi(x_k)V^*(t_1)\ldots V^*(t_m)|0\rangle_{(0)}, \end{aligned} \tag{28}$$

with the interaction potential (26). In the above formula, $H$ indicates the Heisenberg picture and (0) the fact that we use free fields of the Dirac picture on the RHS. For simplicity, we will drop the index (0). It is important to note that time ordering is performed with respect to $x_1^0, \ldots, x_k^0, t_1, \ldots t_m$. The time arguments within a vertex (cf. (26)) are not dissolved (TOPT) [6, 8–10].

The first step is to draw all possible momentum space diagrams with $k$ external legs, as described in [11]. We have to label each line with its 4-momentum $p_i$ including its direction and the variable $\sigma_i$, where $p_\mu^\sigma = (\sigma\omega_p, -\mathbf{p})^{\mathrm{T}}$, $\sigma = \pm 1$. To each line – with labels $p_i$ and $\sigma_i$ – we have to assign the factor

$$\frac{-\mathrm{i}}{p_i^2 + m_i^2 - \mathrm{i}\epsilon}\ \frac{\omega_{p_i} + \sigma_i p_i^0}{2\omega_{p_i}}. \tag{29}$$

The function $\chi$ is associated with each vertex:

$$\begin{aligned} &\chi\left(p_1^{\sigma_1}, \ldots, p_k^{\sigma_k}\right) \\ &= \frac{\lambda}{k!} N^k \int \mathrm{d}^n a_1 \ldots \mathrm{d}^n a_k \mathrm{e}^{-\sum_j a_j^{\mathrm{T}} a_j} \\ &\qquad \times \sum_{Q \in S^k} \exp\left(-\mathrm{i} \sum_j p_j^{\sigma_j} M\, a_{Q_j}\right) \\ &= \lambda \exp\left(-\frac{1}{4} \sum_i p_i^{\sigma_i \mathrm{T}} \kappa\, p_i^{\sigma_i}\right), \end{aligned} \tag{30}$$

where we have summed over all permutations $Q \in S^k$ of the external momenta. By definition, the above integral is independent of the order of the momenta $p_i$. Remarkably, there are only on-shell momenta involved because of (22). We have chosen

$$N = \pi^{-2}.$$

Note that

$$p^{\mathrm{T}} \kappa\, q = p_\mu q_\nu\, \kappa^{\mu\nu} \tag{31}$$

and

$$\kappa^{\mu\nu} = \kappa^{\nu\mu}.$$

Additionally, we have to introduce the usual symmetry factor $\frac{1}{S}$ and to assure momentum conservation at each vertex,

$$(2\pi)^4 \delta^4(p_1 + \ldots + p_k). \tag{32}$$

Finally, we have to integrate over all internal momenta $q_r$ which are not fixed by momentum conservation

$$\prod_{r=1}^{\#\mathrm{Loops}} \frac{\mathrm{d}^4 q_r}{(2\pi)^4} \tag{33}$$

and sum over all $\sigma_i$'s.

As an example, let us consider the contribution of a line between two internal points belonging to different interaction regions ("internal propagator"), i.e. corresponding to different interaction potentials $V(t_i)$ in (28). Therefore, we have to account for a line labelled by $q$ and $\sigma$ and two vertices characterised by $\chi(q^\sigma, \ldots)$ and $\chi(-q^\sigma, \ldots)$, respectively. Sticking everything together yields

$$\begin{aligned} &\Delta_M(x-y) \\ &= \frac{-\mathrm{i}}{(2\pi)^4} \int \mathrm{d}^4 q\, \frac{\mathrm{e}^{-\mathrm{i}q(x-y)}}{q^2 - m^2 + \mathrm{i}\epsilon} \sum_{\sigma=\pm 1} \frac{\omega_q + \sigma q^0}{2\omega_q} \mathrm{e}^{-\frac{1}{2} q^{\sigma \mathrm{T}} \kappa q^\sigma}. \end{aligned} \tag{34}$$

Equation (34) for the "internal propagator" can also be obtained by contracting two smeared field operators (22),

$$\langle 0|T\, \phi_M(x)\phi_M(y)|0\rangle = \Delta_M(x-y). \tag{35}$$

The time ordered product can easily be written as a sum of two terms

$$\begin{aligned} &\langle 0|T\, \phi_M(x)\phi_M(y)|0\rangle \\ &\quad = \langle 0|\phi_M(x)\phi_M(y)|0\rangle\, \theta\left(x^0 - y^0\right) \\ &\qquad + \langle 0|\phi_M(y)\phi_M(x)|0\rangle\, \theta\left(y^0 - x^0\right). \end{aligned} \tag{36}$$

Inserting (22) and the integral representation of the Heaviside step function

$$\theta\left(t' - t\right) = \lim_{\epsilon \to 0} \frac{-1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau}{\tau + \mathrm{i}\epsilon}\, \mathrm{e}^{-\mathrm{i}\tau\left(t'-t\right)} \tag{37}$$

into (36) yields

$$\begin{aligned} &\lim_{\epsilon \to 0} \frac{-1}{2\pi\mathrm{i}} \int \frac{\mathrm{d}^3 k\, \mathrm{d}\tau}{(2\pi)^3\, 2\omega_k}\ \mathrm{e}^{-k_\mu^+ k_\nu^+\, \kappa^{\mu\nu}/2} \\ &\quad \times \left(\mathrm{e}^{-\mathrm{i}\omega_k\left(x^0 - y^0\right) + \mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{y})}\ \frac{\mathrm{e}^{\mathrm{i}\tau\left(x^0 - y^0\right)}}{\tau - \mathrm{i}\epsilon}\right. \\ &\qquad \left. +\ \mathrm{e}^{\mathrm{i}\omega_k\left(x^0 - y^0\right) - \mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{y})}\ \frac{\mathrm{e}^{\mathrm{i}\tau\left(x^0 - y^0\right)}}{\tau + \mathrm{i}\epsilon}\right). \end{aligned} \tag{38}$$

The exponential damping is the only difference from the usual local calculation. After some substitutions and noting that for the substitution $\mathbf{k} \to -\mathbf{k}$ we get $k^+ \to -k^-$ we obtain the desired result:

$$\langle 0|T\, \phi_M(x)\phi_M(y)|0\rangle \tag{39}$$

$$= \frac{-\mathrm{i}}{(2\pi)^4} \int \mathrm{d}^4 q \, \frac{e^{-\mathrm{i}q(x-y)}}{q^2 - m^2 + \mathrm{i}\epsilon} \sum_{\sigma=\pm 1} \frac{\omega_q + \sigma q^0}{2\omega_q} \mathrm{e}^{-\frac{1}{2} q^{\sigma \mathrm{T}} \kappa q^{\sigma}} .$$

Equation (35) allows also for a different interpretation of the Feynman rules. Namely, we can attribute an exponential damping factor

$$\mathrm{e}^{-\frac{1}{2} q^{\sigma \mathrm{T}} \kappa \, q^{\sigma}} \tag{40}$$

to internal lines labelled by $q, \sigma$. The damping can be assigned either to the internal lines or to the vertices. Of course, the amplitudes are unaffected by this choice.

In the situation discussed here, free propagators are not changed, since

$$G(p) = \sum_{\sigma} \frac{-\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \, \frac{\omega_p + \sigma p^0}{2\omega_p} = \frac{-\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}, \tag{41}$$

with $p^0 = \omega_p$ for external particles.

In the next section, we will examine 1-loop corrections and show that they are all finite. Let us first discuss specific choices of the matrix $\kappa$, respectively $M$. For simplicity, we concentrate on the case of a diagonal matrix $\kappa$.

The first choice we want to consider is the unit matrix,

$$(\kappa^{\mu\nu}) = 2\,\zeta \;\; 1. \tag{42}$$

This can be accomplished, for example by using the following matrix $M$:

$$(M^{\mu}{}_{r}) = \sqrt{2\,\zeta} \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} . \tag{43}$$

The motivation to use an antisymmetric matrix $M$ of full rank has already been stressed in [14]. We want to relate this approach to the non-commutativity of space-time. One of the block diagonal matrices of (43) is related to the non-commutative structure in [14], cf. (3) with $\theta = 2\,\zeta$. Explicitly we have

$$p^{+}_{\mu} \kappa^{\mu\nu} p^{+}_{\nu} = 2\zeta \left(\mathbf{p}^2 + \omega_p^2\right) = 2\zeta \left(2\mathbf{p}^2 + m^2\right) , \tag{44}$$

where the second term can be absorbed within the normalisation constant in (22). Therefore, this case is equivalent to choosing $\kappa^{00} = 0$. In general, the case $\kappa^{0i} = 0$ is equivalent to the case $\kappa^{0\mu} = 0$.

The smearing of the field operators considered in the next section will only extend over the spatial dimensions, and the zero component of the 4-vector $Ma$ in (17) vanishes. In this case the Feynman rules become simpler. The factor $\chi$ associated to the vertices becomes

$$\chi \left(p_1^{\sigma_1}, \ldots, p_k^{\sigma_k}\right) = \lambda \, \exp\left( -\frac{1}{4} \sum_i \mathbf{p}_i{}^{\mathrm{T}} \tilde{\kappa} \, \mathbf{p}_i \right) , \tag{45}$$

which only contains the spatial components of the incoming momenta. We will examine the cases

$$\tilde{\kappa} = 2\,\zeta \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} ,$$

$$\tilde{\kappa} = 2\,\zeta \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} , \tag{46}$$

$$\tilde{\kappa} = 2\,\zeta \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} .$$

Therefore, the $\sigma_i$'s are only contained in the contributions assigned to lines, cf. (29). They easily factorise, and

$$\sum_{\sigma_1} \frac{\omega_{q_1} + \sigma_1 q_1^0}{2\omega_{q_1}} \ldots \sum_{\sigma_k} \frac{\omega_{q_k} + \sigma_k q_k^0}{2\omega_{q_k}} = 1. \tag{47}$$

Hence, we have to assign to every line the usual factor

$$\frac{-\mathrm{i}}{p_i^2 - m_i^2 + \mathrm{i}\epsilon}. \tag{48}$$

## 3 Perturbative corrections and power counting

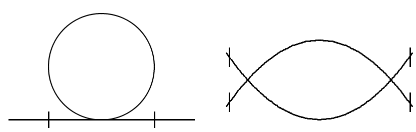
In this section, some properties of perturbative calculations with damped scalar field propagators will be studied. First, we will elaborate a power counting criterion by examining tadpole loops as shown in Figs. 1 and 2. Finally, this criterion will be tested for various calculations in Euclidean as well as Minkowski space.

As indicated in the previous section, the first $j \leq 4 = d$ matrix elements in the diagonal of $\kappa$ are assumed to be 1, whereas all the other elements are assumed to be zero.

For simplicity, the damping factor will be kept track of by putting it into the damped "internal propagator"

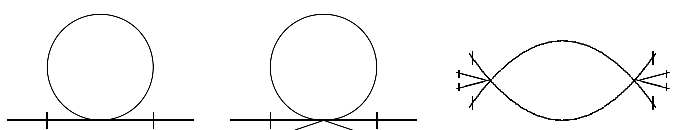
$$\Delta_j(k) \equiv \frac{\mathrm{e}^{-\zeta \sum_{i=1}^{j} k_i^2}}{k^2 - m^2 + \mathrm{i}\,\epsilon} \tag{49}$$

as already indicated by (34). $\zeta$ has dimension of $[\text{length}]^2$, possibly related to the deformation parameter of NCQFT [12–14]. $j$ denotes the number of damped dimensions. The case $j = d$ actually does not fit into our approach of smeared field operators, since the zero component of the occurring



a b

**Fig. 1a,b.** 1-loop contributions for $\phi^4$ theory



a b c

**Fig. 2a–c.** 1-loop graphs

momenta are never involved in the damping factor; cf. (22). Only on-shell momenta occur. However, we will also treat this case in the Euclidean theory since it is manifestly covariant, and it does not lead to much extra work. The definitions

$$\bar{k}^2 \equiv \sum_{i=1}^{j} k_i^2, \tag{50}$$

$$k'^2 \equiv \sum_{i=j+1}^{d} k_i^2 \tag{51}$$

will also be helpful.

### 3.1 Power counting

In order to get a feeling for the power counting behaviour of perturbative calculations with damped internal scalar field propagators in a $d$-dimensional space-time, we present some general statements. Any vertex function is characterised by the number of external lines $E$, the number of internal lines $I$ and the number of interaction vertices $V$.

A general vertex in coordinate space is of the form

$$V_i = \int \mathrm{d}^d x\, \partial_x^{\delta_i} \phi^{b_i}(x), \tag{52}$$

where $\delta_i$ counts the number of derivatives, and $b_i$ stands for the number of scalar fields involved in the interaction.

Let us first consider full damping, i.e. $j = d$. The "internal propagators" described in Sect. 2 are assumed to have the following form in an Euclidean formulation:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{e}^{\mathrm{i}k(x-y)} \frac{1}{k^2+m^2} \mathrm{e}^{-\zeta k^2}, \tag{53}$$

neglecting some factors, which are not important for our considerations here; cf. (39). In momentum space, this involves

$$\varDelta_M(k) = \frac{1}{k^2+m^2}\, \mathrm{e}^{-\zeta k^2}, \tag{54}$$

where $k^2 = \left(k^0\right)^2 + \mathbf{k}^2$. For a fixed $n$, we rewrite (54) as

$$\begin{aligned} \varDelta_M^n(k) &= \frac{1}{(k^2+m^2)\left(1+\zeta k^2+\ldots+\frac{1}{n!}\left(\zeta k^2\right)^n\right)} \\ &= \frac{1}{(\zeta k^2)^n\,(k^2+m^2)\left(1/n! + \mathcal{O}\left(1/\left(\zeta k^2\right)^j\right)\right)}, \end{aligned} \tag{55}$$

with $1 \le j \le n$. In order to estimate the high momentum behaviour of $\varDelta_M^n(k)$ it is sufficient to use

$$\varDelta_M^n(k) \approx \frac{n!}{\left(\zeta k^2\right)^n k^2}. \tag{56}$$

For all high momenta $k$ there is a polynomial $P_n(k^2)$ of degree $n \in \mathbb{N}$ such that

$$\mathrm{e}^{\zeta k^2} > P_n\left(k^2\right) \text{ and } \mathrm{e}^{-\zeta k^2} < \frac{1}{P_n\left(k^2\right)}.$$

The superficial degree of divergence of any vertex graph $\gamma$ is therefore given by

$$D_n(\gamma) = dL - (2n+2)I + \sum_{i=1}^{V} \delta_i. \tag{57}$$

Using

$$L = I - (V-1) \tag{58}$$

and the total number of all lines running to all vertices

$$\sum_i b_i = 2I + E, \tag{59}$$

we get for (57)

$$D_n(\gamma) = d - \dim\phi\, E - \sum_i (d - d_i) - 2n\, I. \tag{60}$$

The dimension of the scalar field is given by

$$\dim\phi = \frac{d}{2} - 1, \tag{61}$$

and the corresponding dimension of the interaction vertex $V_i$ is defined as

$$d_i \equiv \delta_i + \left(\frac{d}{2} - 1\right) b_i. \tag{62}$$

For $n = 0$, we have the usual power counting. Now we are in a position to discuss specific models.

In $d = 3$ space-time dimensions, we have two classical interactions

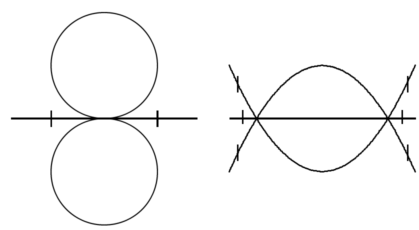
$$V_1^3 = \frac{\lambda_1}{4!} \int \mathrm{d}^3 x\, \phi^4(x) \quad \text{and} \quad V_2^3 = \frac{\lambda_2}{6!} \int \mathrm{d}^3 x\, \phi^6(x). \tag{63}$$

In this case, $\dim\phi = 1/2$. This implies that $\lambda_1$ has dimension of a mass, and $\lambda_2$ is dimensionless. The corresponding analogous interaction of a $\phi^4$-model is ($\dim\phi = 1$)

$$V_3^4 = \frac{\lambda_3}{4!} \int \mathrm{d}^4 x\, \phi^4(x). \tag{64}$$

For $d = 3$, some perturbative corrections up to third order are shown in Figs. 2–4.

According to (60), we can find the degrees of divergence for these classes of radiative corrections; see Table 1.



a b

**Fig. 3a,b.** 2-loop corrections

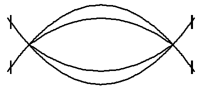


**Fig. 4.** 3-loop correction

**Table 1.** Degrees of divergence as referred to in the text

| Fig. 2a | Fig. 2b |
|---|---|
| $D_n = 1 - 2n < 0,\ \forall n > 0$ | $D_n = 1 - 2n < 0,\ \forall n > 0$ |
| $n = 0 : D_0 = 1$ | $n = 0 : D_0 = 1$ |
| Fig. 2c: finite | |
| Fig. 3a | Fig. 3b |
| $D_n = 2 - 4n < 0,\ \forall n > 0$ | $D_n = -6n < 0,\ \forall n > 0$ |
| $n = 0 : D_0 = 2$ | $n = 0 : D_0 = 0$ |
| Fig. 4 | |
| $D_n = 1 - 8n < 0,\ \forall n > 0$ | |
| $n = 0 : D_0 = 1$ | |

**Table 2.** Degrees of divergence for $d = 4$

| Fig. 1a | Fig. 1b |
|---|---|
| $D_n = 2 - 2n < 0,\ \forall n > 1$ | $D_n = -2n < 0,\ \forall n > 0$ |
| $n = 0 : D_0 = 2$ | $n = 0 : D_0 = 0$ |

For $d = 4$, the corrections at the one loop level are shown in Fig. 1. The degrees of divergence are given by Table 2. Thus, we have finiteness for all above mentioned graphs with fully damped propagators for $n > 1$.

In order to describe the power counting behaviour of the tadpole contribution with a partial damping in some directions in the Euclidean formulation, we have to consider the following integral:

$$\Gamma_{tp}^{j} \equiv \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \ \Delta_j(k) \ . \tag{65}$$

This integral can be rewritten as

$$\Gamma_{tp}^{j} = \frac{1}{(2\pi)^d} \int \mathrm{d}^j \bar{k} \ \mathrm{e}^{-\zeta \bar{k}^2} \int \mathrm{d}^{d-j} k' \ \frac{1}{\bar{k}^2 + k'^2 + m^2} \ . \tag{66}$$

The case $j = d$ has already been discussed. Now, we approximate (66) with a finite parameter $l$ as

$$\Gamma_{tp}^{j,l} = \frac{1}{(2\pi)^d} \int \mathrm{d}^j \bar{k} \ \frac{1}{1 + \ldots + \frac{1}{l!}\left(\zeta \bar{k}^2\right)^l} \times \int \mathrm{d}^{d-j} k' \ \frac{1}{\bar{k}^2 + k'^2 + m^2} \ . \tag{67}$$

For $0 < j \leq d$, there exists always a $l > 0$ such that the $\bar{k}$-integration converges. It remains to estimate the $k'$-integration. Naive power counting can be applied. For the tadpole, we get

$$D_j = (d - j) - 2, \tag{68}$$

For $d = 3$ and $j = 2$, one has $D_j = -1$. This will be checked by explicit calculations in Sect. 3.1.

For $d = 4$, we conclude from (68) that the degree of damping has to be $j > 2$ in order to have convergence. We will see that these results are compatible with direct calculations.

Using the same philosophy, we can discuss an arbitrary $L$-loop contribution. We can estimate the naive power counting (assuming that the integration over the $j$ damped directions is convergent) by

$$D_j = L(d - j) - 2I + \sum_i \delta_i, \tag{69}$$

implying

$$D_j = d - \dim\phi\, E - \sum_i (d - d_i) - jL. \tag{70}$$

Equation (70) seems to imply that the superficial degree of divergence $D_j$ linearly decreses with the number of loops $L$. But $L$ and the number of vertices are related. We can rewrite (70) in the following way:

$$D_j = d - j - E\,\dim\phi + \frac{Ej}{2} - \sum_i \left(d - \delta_i - b_i\left(\frac{d}{2} - 1\right)\right) - \sum_i \left(\frac{b_i}{2} - 1\right) j. \tag{71}$$

We see that $D_j$ decreases with the number of vertices and may increase with the number of external legs.

For $j = 0$ (no damping), we get back the power counting behaviour of a local theory.

As a further consistency check, we discuss (70) for the tadpole contribution with $L = 1$. For $d = 3$, we have the following: $E = 2$ and $d - d_i = 1$, for the $\phi^4$ interaction; $E = 4$ and $d - d_i = 0$, for the $\phi^6$ interaction. Therefore, both cases yield

$$D_j = 1 - j. \tag{72}$$

This implies convergence for $j > 1$.

For $d = 4$, we find

$$D_j = 2 - j, \tag{73}$$

meaning that convergence implies $j > 2$. The fact that the degree of divergence depends on the number of smeared dimension has also been observed in [17] where the vacuum energy density has been discussed in the framework of the third approach of Sect. 1.

### 3.2 Explicit calculations in the Euclidean case

Let us consider the tadpole integral

$$\Gamma_{tp}^{j} = (2\pi)^{-d} \int \mathrm{d}^d k \ \frac{\mathrm{e}^{-\zeta \bar{k}^2}}{k^2 + m^2} \tag{74}$$

in $d = 3$ and 4 dimensions. In 3 space-time dimensions, we have to solve the following integral:

$$\Gamma_{tp}^{2} = (2\pi)^{-3} \int \mathrm{d}^3 k \ \frac{\mathrm{e}^{-\zeta \bar{k}^2}}{k^2 + m^2}. \tag{75}$$

The relevant loop graphs are Figs. 2a and 3a. We employ the Schwinger parametrisation

$$\frac{1}{k^2+m^2} = \int_0^\infty \mathrm{d}\alpha \mathrm{e}^{-\alpha\left(k^2+m^2\right)} \tag{76}$$

to obtain

$$\begin{aligned} \Gamma_{tp}^2 &= \frac{\pi^{3/2}}{(2\pi)^3}\int_0^\infty \frac{1}{\alpha^{1/2}(\alpha+\zeta)}\mathrm{e}^{-\alpha m^2} \\ &= \frac{\pi^{3/2}}{(2\pi)^3}\,\mathrm{e}^{\zeta\, m^2}\sqrt{\frac{\pi}{\zeta}}\;\Gamma\left(1/2,\zeta\, m^2\right). \end{aligned} \tag{77}$$

$\Gamma(z)$ is the ordinary Gamma function, whereas $\Gamma(1/2, z^2)$ denotes the "finite" incomplete Gamma function,

$$\Gamma\left(1/2,\zeta\, m^2\right) = \sqrt{\pi} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1/2)}\left(\zeta\, m^2\right)^{n+1/2}. \tag{78}$$

For $j = 1$, the tadpole contribution diverges.

In 4 dimensions, the calculations are a bit more involved. The tadpole integral is given by

$$\Gamma_{tp}^j = (2\pi)^{-4}\int \mathrm{d}^4k\;\frac{\mathrm{e}^{-\zeta\bar{k}^2}}{k^2+m^2}. \tag{79}$$

But already at this step, it is clear that UV/IR mixing as it occurs in the first approach of Sect. 1 does not appear for the tadpole here. UV/IR mixing basically means that divergences due to UV-integrations arise for vanishing external momenta. But here, the integration does not even depend on the external momenta. Using the Schwinger parametrisation and carrying out the Gaussian integration, we obtain

$$\Gamma_{tp}^j = \frac{\pi^2}{(2\pi)^4}\int_0^\infty \mathrm{d}\alpha\;\frac{\mathrm{e}^{-\alpha m^2}}{\alpha^{2-j/2}(\alpha+\zeta)^j/2}. \tag{80}$$

The possible problems of UV-integration are now hidden in the behaviour of this integral for $\alpha \to 0$. The power counting behaviour can be studied by regulating this expression which is done by restricting the integration to $\alpha \in [1/\Lambda^2, \infty[$. By dividing this area of integration into $[1/\Lambda^2, a[$ and $[a, \infty[$ with $a \ll \zeta$, we can read off the degree of divergence to be

$$D_j = 2 - j. \tag{81}$$

This agrees with the power counting formula given above and tells us that the tadpole is quadratically, linearly and logarithmically divergent for $j = 0, 1, 2$, respectively. The minimum damping rendering the tadpole contribution finite is given by $j = 3$. We obtain

$$\Gamma_{tp}^3 = \frac{2\pi^{5/2}}{(2\pi)^4\zeta}U\left(\frac{1}{2}, 0, m^2\zeta\right), \tag{82}$$

where $U$ denotes the confluent hypergeometric function, with $U\left(\frac{1}{2}, 0, 0\right) = \frac{2}{\sqrt{\pi}}$. Of course, also the case $j = 4$ gives a finite result:

$$\Gamma_{tp}^4 = \frac{\pi^2}{(2\pi)^4}\left[\zeta^{-1} + m^2\mathrm{e}^{\zeta m^2}\mathrm{Ei}\left(-\zeta m^2\right)\right], \tag{83}$$

where Ei is the exponential integral function with the following expansion for $x < 0$:

$$\mathrm{Ei}(x) = \gamma + \ln(-x) + \sum_{k=1}^{\infty}\frac{x^k}{k(k!)}, \tag{84}$$

where $\gamma$ is the Euler constant. Note that the parameter $\zeta$ acts as a regulator. For any $j$, the integral (79) diverges quadratically for vanishing $\zeta$.

In a second step, we consider the one loop 4-point 1PI vertex (see Fig. 1b). The corresponding Feynman integral is

$$\Gamma_4^j(p) \equiv \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\;\Delta_j(k)\Delta_j(k+p). \tag{85}$$

Using the two Schwinger parameters $\alpha$, $\beta$ and applying the integral transformation

$$\begin{aligned} \alpha &= (1-\xi)\lambda, \\ \beta &= \xi\;\lambda, \end{aligned} \tag{86}$$

we get

$$\begin{aligned} \Gamma_4^j(p) &= \frac{\pi^2}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda\int_0^1 \mathrm{d}\xi\,\frac{1}{\lambda^{1-j/2}(\lambda+2\zeta)^{j/2}} \\ &\times \mathrm{e}^{-\sum_{i=1}^j p_i^2\frac{\xi(1-\xi)\lambda^2+\zeta(\lambda+\zeta)}{\lambda+2\zeta}-\left(p^2-p_i^2\right)\xi(1-\xi)\lambda-\lambda m^2}. \end{aligned} \tag{87}$$

A further evaluation of these integral is quite tricky. But the UV-behaviour can again be read off from the properties of the denominator:

$$\lambda^{1-j/2}(\lambda+2\zeta)^{j/2},$$

for $\lambda \to 0$. The only problems might arise from the first factor and we do not expect UV-divergences for $1-j/2 < 1$ or $j > 0$. This means that at least one direction of space-time has to be damped in order to render the integral $\Gamma_4^j$ finite, which again agrees with our power counting criterion $D_j = -j < 0$. A more detailed analysis of the integral was only possible for $j = 4$, where it could be rewritten after an appropriate transformation as

$$\Gamma_4^j(p) = -\frac{\pi^2}{(2\pi)^4}\mathrm{e}^{2\zeta m^2}\int_0^1 \mathrm{d}\xi\,\mathrm{Ei}\left(-2\left[\xi(1-\xi)p^2+m^2\right]\zeta\right). \tag{88}$$

This expression is finite since Ei in the integrand is evaluated at negative values only, where it is well behaved, and the integral itself is over a finite interval.

### 3.3 Explicit calculations in the Minkowski case

Now we are ready to carry out a similar analysis for Minkowski space. The tadpole diagram corresponds to the integral

$$\Gamma_{tp}^j \equiv \int \mathrm{d}^4k\;\frac{\mathrm{e}^{-\zeta\bar{k}^2}}{k^2-m^2+\mathrm{i}\epsilon}. \tag{89}$$

The case of full damping (in all space-time directions) is omitted for Minkowski space, where we would have to use

$\exp[-\zeta(k_1^2+k_2^2+k_3^2+k_0^2)]$ as a damping factor to ensure finiteness. Wick rotation is not possible for the fully damped Minkowski situation since one would encounter exploding factors $\exp(-\zeta k_0^2) \to \exp(\zeta k_4^2)$. For the following discussion, the exponential is assumed not to depend on $k_0$. Hence, there are no obstacles opposing Wick rotation, and the results of the preceding discussion in Euclidean space for $j<4$ are directly applicable.

We now turn to the more complicated kind of loops as shown in Fig. 1. The interesting part of this diagram is given by the integral

$$\begin{aligned} &\Gamma_4^j(p=p_1+p_2) \\ &\equiv \int \mathrm{d}^4k\ \frac{\mathrm{e}^{-\zeta\bar{k}^2}}{k^2-m^2+\mathrm{i}\epsilon}\,\frac{\mathrm{e}^{-\zeta\sum_i(k+p)_i^2}}{(k+p)^2-m^2+\mathrm{i}\epsilon} \\ &\equiv \int \mathrm{d}^4k\ f(\mathbf{k},\mathbf{p})\,g(k,p), \end{aligned} \tag{90}$$

where

$$f(\mathbf{k},\mathbf{p}) \equiv \mathrm{e}^{-\zeta\bar{k}^2}\mathrm{e}^{-\zeta\sum_i(k+p)_i^2}.$$

$f$ only depends on the spatial momentum components and not on their time component. The direct evaluation of $\Gamma_4^j$ for arbitrary external momenta $p$ seems to be rather tricky, and here we restrict ourselves to the UV-behaviour concerning the $k$ integration. We want to give an upper bound for $\Gamma_4^j$ and show that it is finite. But let us first get rid of the poles concerning the $k^0$ integration. This is most easily accomplished by the residue theorem

$$I^0(\mathbf{k},p) \equiv \int \mathrm{d}k^0\ g(k,p) = \pi\mathrm{i}\frac{\left(\frac{1}{\omega_{\mathbf{k}}}+\frac{1}{\omega_{\mathbf{k}+\mathbf{p}}}\right)}{\left(\omega_{\mathbf{k}}+\omega_{\mathbf{k}+\mathbf{p}}\right)^2-{p^0}^2}. \tag{91}$$

The loop integral then reads

$$\Gamma_4^j(p) = \int \mathrm{d}^3k\ f(\mathbf{k},\mathbf{p})\,I^0(\mathbf{k},p).$$

$I^0$ has the following bound:

$$|I^0(\mathbf{k},p)| \le \frac{C}{|\mathbf{k}|^3} \qquad \text{for} \quad |\mathbf{k}| \ge r_p,$$

where $C>0$ is some proportionality constant and

$$r_p \propto \max\left(|p_0|,|\mathbf{p}|\right).$$

Defining the UV-part of the integration $UV \equiv \{\mathbf{k}\in R^3|\ |\mathbf{k}|\ge r_p\}$, we thus conclude

$$\begin{aligned} &|I^{UV}(p)| \equiv |\int_{UV} \mathrm{d}^3k\ f(\mathbf{k},\mathbf{p})\,I^0(\mathbf{k},p)| \\ &\le \int_{UV} \mathrm{d}^3k\ |f(\mathbf{k},\mathbf{p})\,I^0(\mathbf{k},p)| \le \int_{UV} \mathrm{d}^3k\ f(\mathbf{k},\mathbf{p})\,\frac{C}{|\mathbf{k}|^3}. \end{aligned}$$

This is finite as long as the sum over $i$ within $f$ involves at least one of the three spatial components, say $j>0$. This is consistent with the results of the Euclidean discussion, where we concluded to the same UV-behaviour by inspection of (87) for $\lambda \to 0$. It again confirms our power counting criterion.

## 4 Conclusion and remarks

We have discussed a non-local real scalar field theory. The non-locality is located in the interaction where we have replaced the usual local fields by smeared field operators (17). The Feynman rules are worked out in Sect. 2 using the Gell-Mann–Low formula (28). The free theory is not modified. Therefore, also the free propagators are unaltered. As a result of the smearing, the vertex contribution is exponentially damped by the incoming on-shell momenta (30). The fact that on-shell momenta enter the vertex contribution is of vital importance and a natural consequence of TOPT. In contrast to this result, the exponentially damped propagators obtained in [12–14] contain arbitrary momenta.

In Sect. 3, we have carefully discussed UV properties of the model. We have derived a power counting formula (55) which provides the superficial degree of divergence for theories with exponential damping in arbitrarily many dimensions. Explicit calculations of 1-loop diagrams in the Euclidean and Minkowski framework, done in Sect. 3.2 and respectively 3.3, agree with the result from the generalised power counting formula. In $d=3$ space-time dimensions, the tadpole contribution shown in Fig. 2 is finite if at least one dimension is damped, i.e. $j>1$. The other loop contribution in Fig. 1a is finite independently of $j$. In $d=4$ space-time dimensions, the tadpole contribution converges for $j>2$ and the 1-PI graph of Fig. 1b for $j\ge 1$. The power counting formula shows that the presented model is UV finite to all orders in perturbation theory according to the proposed power counting formula. Notably, there is also no UV/IR mixing present at the 1-loop level.

Applying the methods presented here to gauge theories is the next interesting step and may provide new insights.

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## References

1. A. Pais, Phys. Rev. **79**, 145 (1950)
2. H.C. Ohanian, Phys. Rev. D **55**, 5140 (1997); **60**, 104051 (1999)
3. Th. Filk, Phys. Lett. B **376**, 53 (1996)
4. S. Minwalla, M. van Raamsdonk, N. Seiberg, JHEP **0002**, 020 (2000) [hep-th/9912072];
   A. Matusis, L. Susskind, N. Toumbas, JHEP **0012**, 002 (2000) [hep-th/0002075]
5. J. Gomis, Th. Mehen, Nucl. Phys. B **591**, 265 (2000) [hep-th/0005129]
6. D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, Phys. Lett. B **533**, 178 (2002) [hep-th/0201222]
7. Y. Liao, K. Sibold, Eur. Phys. J. C **25**, 469 (2002) [hep-th/0205269]
8. Y. Liao, K. Sibold, Eur. Phys. J. C **25**, 479 (2002) [hep-th/0206011]

9. H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda, R. Wulkenhaar, Eur. Phys. J. C **29**, 133 (2003) [hep-th/0209253]
10. P. Fischer, V. Putz, Eur. Phys. J. C **32**, 269 (2004) [hep-th/0306099]
11. S. Denk, M. Schweda, JHEP **0309**, 032 (2003) [hep-th/0306101]
12. M. Chaichian, A. Demichev, P. Prešnajder, Nucl. Phys. B **567**, 360 (2000) [hep-th/9812180]
13. S. Cho, R. Hinterding, J. Madore, H. Steinacker, Int. J. Mod. Phys. D **9**, 161 (2000) [hep-th/9903239]
14. A. Smailagic, E. Spallucci, J. Phys. A **36**, L517 (2003) [hep-th/0308193]
15. R.J. Glauber, Phys. Rev. **131**, 2766 (1963)
16. D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, Comm. Math. Phys. **237**, 221 (2003) [hep-th/0301100]
17. P. Nicolini, Vacuum energy momentum tensor in $(2+1)$ NC scalar field theory, hep-th/0401204